MATH 5061 Lecture 6 (Feb 24)

Recall: A Riemannian manifold (M^n, g)

where M^n = smooth n-dim'l manifold

For $p \in M$, $g_p := \langle \, , \, \rangle_p$ inner product on the vector space T_pM

$\rightsquigarrow$ concept of "length" & "angles" on each T_pM

Thm: Given (M^n, g), $\exists !$ Levi-Civita connection ∇ s.t.

(1) $X(\langle Y, Z \rangle) = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$ where $X, Y, Z \in \Gamma(TM)$

(2) $\nabla_X Y - \nabla_Y X = [X, Y]$

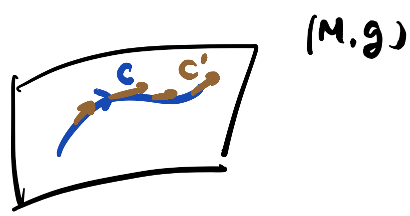
In local coord. $(x^1, \dots, x^n)$ on M, write $\partial_i := \frac{\partial}{\partial x^i}$

$g_{ij}^{(p)} = \langle \partial_i, \partial_j \rangle_p$, (g_{ij}) : $n \times n$ symm pos. definite matrix $\rightsquigarrow$ inverse matrix (g^{ij})

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k \xrightarrow{(1) \& (2)} \Gamma_{ij}^k = \frac{1}{2} g^{k\ell} (\partial_i g_{\ell j} + \partial_j g_{i\ell} - \partial_\ell g_{ij})$$

Note: $\Gamma_{ij}^k = F(g, \partial g)$.

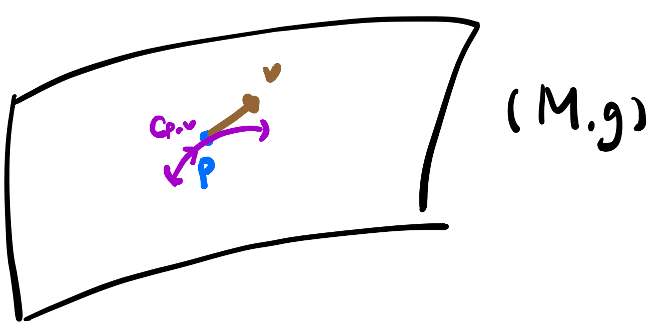
Geodesic eq^n: $\nabla_{c'} c' \equiv 0$



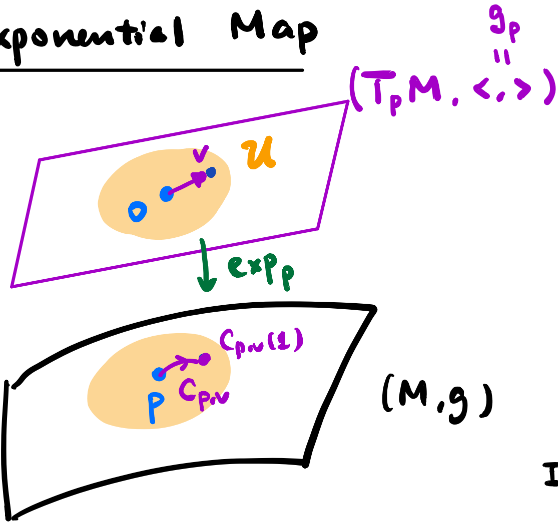
local coord: $$\frac{d^2 c^k}{dt^2} + \Gamma_{ij}^k(c(t)) \frac{dc^i}{dt} \frac{dc^j}{dt} = 0$$

$\xRightarrow[\text{theory}]{\text{ODE}}$ For any fixed $p \in M$, $v \in T_pM$,

$\exists$ "!" geodesic $c_{p,v} : (-\varepsilon, \varepsilon) \to M$ s.t. $c(0) = p$, $c'(0) = v$



Exponential Map



$\exp_p : \underset{0 \in}{\mathcal{U}} \subseteq T_pM \longrightarrow M$

$\exp_p(v) := C_{p,v}(1)$

- $\exp_p(0) = p$
- $d(\exp_p)_0 = id_{T_pM}$

I.F.T.

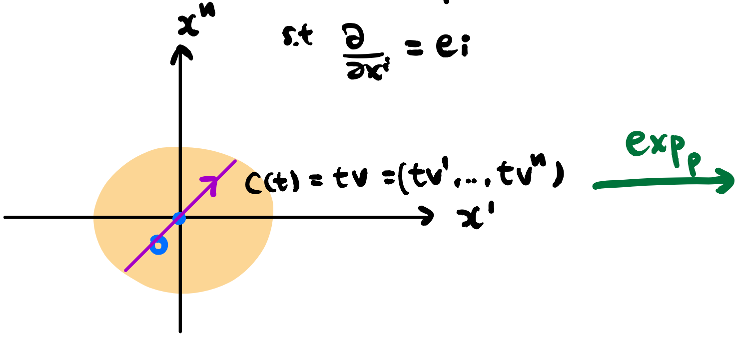
$\Longrightarrow$ $\exp_p$ is a local diffeomorphism near 0 to a nbd. of p.

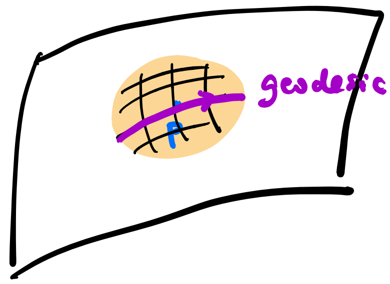
$\rightsquigarrow$ local coordinate system near p called "Geodesic Normal Coordinates"

$\langle , \rangle_p$

$\boxed{T_pM}$ Fix O.N.B $e_1, \dots, e_n$ and choose coord $(x^1, \dots, x^n)$ on T_pM s.t $\frac{\partial}{\partial x^i} = e_i$

$\boxed{M, g}$





Prop: In geodesic normal coord. at $p \in M$,

$$g_{ij}(0) = \delta_{ij} \quad \text{and} \quad \Gamma^k_{ij}(0) = 0$$

(i.e. (M^n, g) "=" $(\mathbb{R}^n, g_{Eucl.})$ at any pt. $\hookrightarrow$ 1st order information at a pt is NOT "Geometric" (indep. of choice of coord.))

Proof: $g_{ij}(0) = \langle \partial_i, \partial_j \rangle_0 = \delta_{ij}$ by construction.

radial lines from 0 $c(t) = tv$ corr. to geodesics on (M, g)

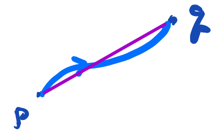
$$\Rightarrow \underbrace{\frac{d^2c^k}{dt^2}}_{=0} + \Gamma^k_{ij} \underbrace{\frac{dc^i}{dt}}_{v^i} \underbrace{\frac{dc^j}{dt}}_{v^j} = 0 \Rightarrow \Gamma^k_{ij}(0) v^i v^j = 0 \;\; \forall v \Rightarrow \Gamma^k_{ij}(0) = 0$$

$\blacksquare$

Recall: "geodesics" $\approx$ "straight lines"

acceleration $= 0$

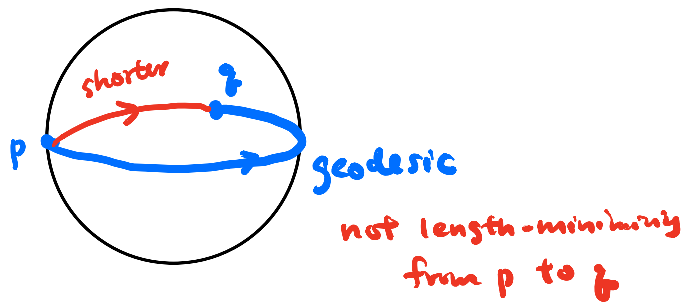
length-minimizing curves



We will see that geodesics are "locally" length-minimizing.

E.g.) (S^2, g_{round})

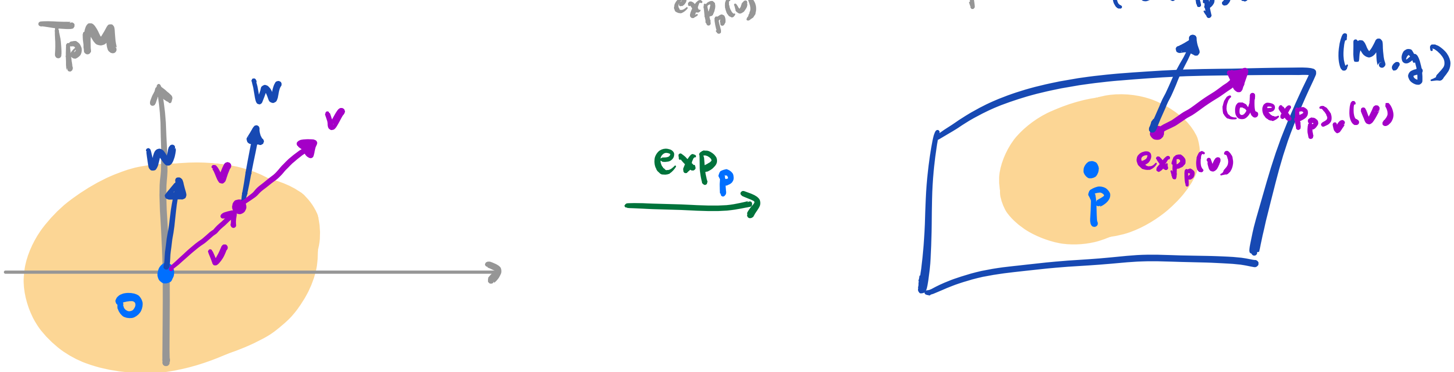
"Long" geodesics are not necessarily minimizing.



Gauss lemma: Let $p \in M$, $v \in T_pM$ s.t. $\exp_p(v)$ is defined.

Then, $\forall\, w \in T_pM \ (\cong T_v(T_pM))$.

$$\langle (d\exp_p)_v(v), (d\exp_p)_v(w) \rangle_{\exp_p(v)} = \langle v, w \rangle_p$$



Proof: **Case 1:** $w = v$

$\because\ t \longmapsto \exp_p(tv)$ geodesic on M

$\Rightarrow$ constant speed $= \|v\|$ at $t = 0$.

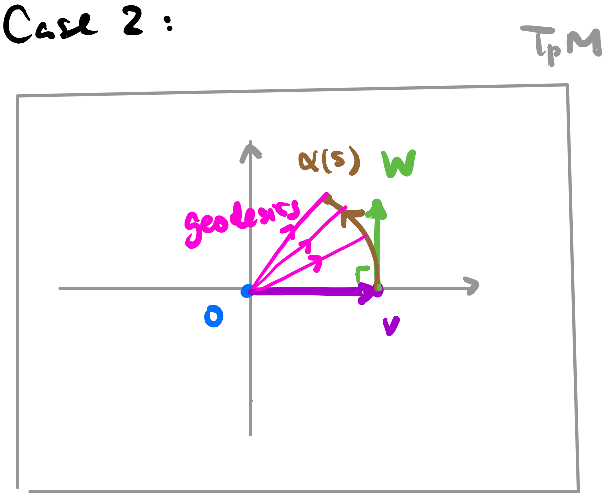
Case 2: $w \perp v$ w.r.t $\langle , \rangle_p$.

It suffices to show

$(d\exp_p)_v(w) \perp (d\exp_p)_v(v)$

w.r.t. $\langle , \rangle_{\exp_p(v)}$

Case 2:



$\alpha(s)$: a curve on T_pM

s.t. $\alpha(0) = v$; $\alpha'(0) = w$

and $\|\alpha(s)\| \equiv \|v\|$

$\rightsquigarrow$ get a 1-parameter family of geodesics by

$$f_s(t) := \exp_p(t\alpha(s)).$$

<u>Note</u>: $f_s(\cdot)$ is a geodesic for each s.

<u>Observe</u>:

$$(d\exp_p)_v(v) = \left.\frac{\partial f}{\partial t}\right|_{t=1,\, s=0}$$

$$(d\exp_p)_v(w) = \left.\frac{\partial f}{\partial s}\right|_{t=1,\, s=0}$$

Consider

$$\frac{\partial}{\partial t}\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle \overset{\substack{\text{metric}\\ \text{compatible}}}{=} \left\langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle + \left\langle \frac{\partial f}{\partial s}, \cancelto{0}{\nabla_{\frac{\partial}{\partial t}} \frac{\partial f}{\partial t}}\right\rangle$$

$$\overset{\substack{\text{torsion}\\ \text{free}}}{=} \left\langle \nabla_{\frac{\partial}{\partial s}} \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right\rangle \overset{\substack{\text{metric}\\ \text{compatible}}}{=} \frac{1}{2}\frac{\partial}{\partial s}\underbrace{\left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right\rangle}_{\substack{\equiv \text{ const. in } s\\ \because \|\alpha(s)\| \equiv \|v\|.}} = 0$$

So, $\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle$ is indep. of t.

$$\left.\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle\right|_{t=1,\, s=0} = \left.\left\langle \underbrace{\frac{\partial f}{\partial s}}_{\substack{= \\ 0}\ \text{at } t=0}, \frac{\partial f}{\partial t}\right\rangle\right|_{t=0,\, s=0} = 0$$

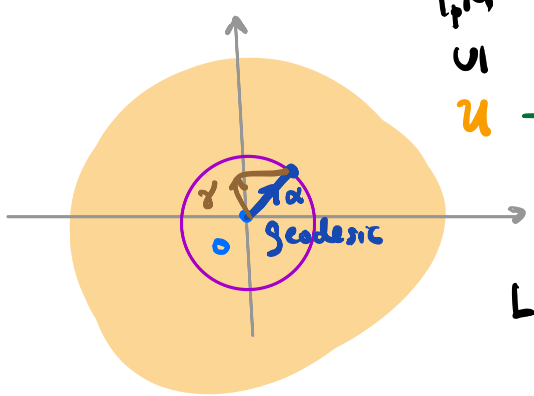
$\square$

Gauss Lemma, in geodesic normal coord., using polar coord (r, θ)

$$\Rightarrow \quad g_{rr} \equiv 1 \quad \text{and} \quad g_{r\theta} \equiv 0$$

$\Rightarrow$ geodesics are locally length-minimizing.

Why?



$$T_pM \supseteq \mathcal{U} \xrightarrow[\text{diffeo.}]{\exp_p} \exp_p \mathcal{U} \subseteq M$$

$$\text{Length}(\gamma) := \int_0^1 \sqrt{\langle \gamma', \gamma' \rangle}\, dt$$

$$= \int_0^1 \sqrt{\underbrace{g_{rr}}_{\substack{= \text{Gauss lemma} \\ 1}} (r')^2 + \underbrace{g_{\theta\theta} (\theta')^2}_{\geq 0} + 2 \underbrace{g_{r\theta}}_{\substack{= \text{Gauss lemma} \\ 0}} r'\theta'}\, dt$$

$$\geq \int_0^1 |r'| dt \geq \int_0^1 r' dt$$

$$= r(1) - r(0) = \text{Length}(\alpha)$$

Riemannian Curvature Tensor

Motivation: $S^2 \subseteq \mathbb{R}^3 \rightsquigarrow$ Gauss curvature K "intrinsic" & Mean curvature H "extrinsic"

Q: What is the "appropriate" notion of curvature for (M, g)?

Note: "higher dim" & "intrinsic".

A: Riem. curvature tensor. Riem. $= R$

Def^{n}: The Riemann curvature of (M^n, g) is an association to each $X, Y \in \mathcal{T}(TM)$ a map

$$R(X, Y) : \mathcal{T}(TM) \longrightarrow \mathcal{T}(TM)$$

defined by

$$R(X,Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

Levi-Civita connection

<u>Remark:</u> $R(X,Y)Z$ are linear in X, Y and Z.

<u>Prop:</u> $R(X,Y)Z$ are "tensorial" in X, Y and Z.

i.e. $R(fX,Y)Z = R(X,fY)Z = R(X,Y)(fZ) = f\,R(X,Y)Z$
$\forall f \in C^\infty(M)$

<u>Proof:</u> Note: $R(X,Y) = -R(Y,X)$, since $[Y,X] = -[X,Y]$.

So it suffice to show $R(fX,Y)Z = f(R(X,Y)Z)$,

and $R(X,Y)(fZ) = f(R(X,Y)Z)$.

<u>$R(fX,Y)Z = f\,R(X,Y)Z$:</u>

$$\begin{aligned}
R(fX,Y)Z &= \nabla_Y \nabla_{fX} Z - \nabla_{fX} \nabla_Y Z + \nabla_{[fX,Y]} Z \\
&= \nabla_Y (f \nabla_X Z) - f \nabla_X \nabla_Y Z + \nabla_{f[X,Y] - Y(f)X} Z \\
&= f \nabla_Y \nabla_X Z + \cancel{Y(f) \nabla_X Z} - f \nabla_X \nabla_Y Z \\
&\quad + f \nabla_{[X,Y]} Z - \cancel{Y(f) \nabla_X Z} \\
&= f\,R(X,Y)Z
\end{aligned}$$

<u>$R(X,Y)(fZ) = f\,R(X,Y)Z$:</u>

$$R(X,Y)(fZ) = \nabla_Y \nabla_X (fZ) - \nabla_X \nabla_Y (fZ) + \nabla_{[X,Y]}(fZ)$$

$$\nabla_Y \nabla_X (fZ) = \nabla_Y \left(f \nabla_X Z + X(f) Z \right)$$

$[Y,X](f)\,Z$
$\|$

$$
\begin{aligned}
\text{so} \quad \nabla_Y \nabla_X (fZ) &= f \nabla_Y \nabla_X Z - \cancel{Y(f) \nabla_X Z} + \cancel{X(f) \nabla_Y Z} + \boxed{\cancel{YX(f)\, Z}} \\
- \quad \nabla_X \nabla_Y (fZ) &= f \nabla_X \nabla_Y Z - \cancel{X(f) \nabla_Y Z} + \cancel{Y(f) \nabla_X Z} + \boxed{\cancel{XY(f)\, Z}} \\
+ \quad \nabla_{[X,Y]} (fZ) &= \boxed{\cancel{[X,Y](f)\, Z}} + f \nabla_{[X,Y]} Z
\end{aligned}
$$

$$\text{R.H.S.} = f R(X,Y) Z$$

$\square$

Def$^{\underline{n}}$: $R(X,Y,Z,W) := \langle R(X,Y)Z, W \rangle$ (0,4)-tensor Riem. curvature tensor

Prop: (Symmetries of Riem. curvature tensor)

(a) Bianchi identity:

$$R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$$

cylic permutation

(b) $R(X,Y,Z,W) = -R(Y,X,Z,W)$

(c) $R(X,Y,Z,W) = -R(X,Y,W,Z)$

(d) $R(X,Y,Z,W) = R(Z,W,X,Y)$

Proof: (a)

$$
\begin{aligned}
R(X,Y)Z &= \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z \\
+ \quad R(Y,Z)X &= \nabla_Z \nabla_Y X - \nabla_Y \nabla_Z X + \nabla_{[Y,Z]} X \\
+ \quad R(Z,X)Y &= \nabla_X \nabla_Z Y - \nabla_Z \nabla_X Y + \nabla_{[Z,X]} Y
\end{aligned}
$$

$$
\begin{aligned}
\text{R.H.S.} = {} & \nabla_Y [X,Z] + \nabla_Z [Y,X] + \nabla_X [Z,Y] \\
& - \nabla_{[X,Z]} Y - \nabla_{[Y,X]} Z - \nabla_{[Z,Y]} X
\end{aligned}
$$

$$= [Y,[X,Z]] + [Z,[Y,X]] + [X,[Z,Y]] = 0$$

"Jacobi identity"

(b) $R(X,Y)Z = -R(Y,X)Z$. done!

(c) It suffices to show $R(X,Y,T,T)=0$ ($\because$ set $T = W+Z$)

$$\begin{aligned} R(X,Y,T,T) &= \langle R(X,Y)T, T\rangle \\ &= \langle \nabla_Y\nabla_X T - \nabla_X\nabla_Y T + \nabla_{[X,Y]}T, T\rangle \\ &= \langle \nabla_Y\nabla_X T, T\rangle - \langle \nabla_X\nabla_Y T, T\rangle + \langle \nabla_{[X,Y]}T, T\rangle \end{aligned}$$

Note:

$$\langle \nabla_Y\nabla_X T, T\rangle = Y\langle \nabla_X T, T\rangle - \cancel{\langle \nabla_X T, \nabla_Y T\rangle}$$

$$- \quad \langle \nabla_X\nabla_Y T, T\rangle = X\langle \nabla_Y T, T\rangle - \cancel{\langle \nabla_Y T, \nabla_X T\rangle}$$

$$+ \quad \langle \nabla_{[X,Y]}T, T\rangle = \frac{1}{2}[X,Y]\langle T,T\rangle$$

$$\text{R.H.S.} = Y\left(\tfrac{1}{2}X\langle T,T\rangle\right) - X\left(\tfrac{1}{2}Y\langle T,T\rangle\right) + \tfrac{1}{2}[X,Y]\langle T,T\rangle$$

$$= 0$$

(d) Bianchi $\Rightarrow$

$$R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$$

$$+ \quad R(Y,Z,W,X) + R(Z,W,Y,X) + R(W,Y,Z,X) = 0$$

$$+ \quad R(Z,W,X,Y) + R(W,X,Z,Y) + R(X,Z,W,Y) = 0$$

$$+ \quad R(W,X,Y,Z) + R(X,Y,W,Z) + R(Y,W,X,Z) = 0$$

Cancels

$$2\left(R(Z,X,Y,W) + R(W,Y,Z,X)\right) = 0$$

$$\Rightarrow R(Z,X,Y,W) = R(Y,W,Z,X)$$

$\square$